Key

Math 4 Name ________________________

4-6 Sum and Difference Formulas for Cosine

Date________

In this activity, you will be working towards the following learning goal:
I can, without a calculator, use trigonometric identities such as angle addition/subtraction and double angle formulas, to express values of trigonometric functions in terms of rational numbers and radicals

I. **Evaluate the following with your calculator to 4 decimal places:** $\cos 105° \approx$ -0.2588

The problem is that this is not an exact value because **cos 105°** is an irrational number.
The following formulas will enable you to calculate the exact value of **cos 105°**.

Sum and Difference Formulas for Cosine:

$$\cos(\alpha \pm \beta) = \begin{cases} \cos(\alpha + \beta) = \cos\alpha \bullet \cos\beta - \sin\alpha \bullet \sin\beta \\ \cos(\alpha - \beta) = \cos\alpha \bullet \cos\beta + \sin\alpha \bullet \sin\beta \end{cases}$$

Note: α and β are angle measures in either degrees or radians.

Here's how they're used . . .

1. Think of **cos 105°** as **cos(45° + 60°)**. What is α? 45° What is β? 60°
2. Use: $\cos(\alpha + \beta) = \cos\alpha \bullet \cos\beta - \sin\alpha \bullet \sin\beta$
3. Substitute: $\cos(45 + 60) = \cos 45 \bullet \cos 60 - \sin 45 \bullet \sin 60$
4. Now it's *Happy Unit Circle Time* . . . . Evaluate and simplify. Combine into one fraction. DO NOT TOUCH YOUR CALCULATOR.

$$\cos(45° + 60°) = \cos 45° \bullet \cos 60° - \sin 45° \bullet \sin 60°$$

$$= \frac{\sqrt{2}}{2} \bullet \frac{1}{2} - \frac{\sqrt{2}}{2} \bullet \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{\boxed{\sqrt{2}} - \boxed{\sqrt{6}}}{\boxed{4}}$$

5. *Now use your calculator.* Enter your final fraction from step 4 into your calculator and evaluate. What do you get? -0.2588

For α and β values, you must use those from the $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ families.

CAUTION!

$\cos(\alpha + \beta) \neq \cos\alpha + \cos\beta$

$\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$

Example

We know, from our knowledge of the unit circle, that $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$. *We can use this knowledge to verify the angle addition formula. See below.*

$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{8\pi}{6} + \frac{3\pi}{6}\right) = \cos\left(\frac{4\pi}{3} + \frac{\pi}{2}\right).$$

Using our angle addition formula with $\alpha = \frac{4\pi}{3}$ and $\beta = \frac{\pi}{2}$

$$\cos\left(\frac{4\pi}{3} + \frac{\pi}{2}\right) = \cos\left(\frac{4\pi}{3}\right) \cdot \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{4\pi}{3}\right) \cdot \sin\left(\frac{\pi}{2}\right) = -\frac{1}{2} \cdot 0 - \left(-\frac{\sqrt{3}}{2}\right) \cdot 1 = \boxed{\frac{\sqrt{3}}{2}}$$

Now you try. Find the exact values for the following.

1. $\cos\left(\frac{11\pi}{12}\right) = \cos\left(\frac{9\pi}{12} + \frac{2\pi}{12}\right) = \cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

$= \cos\frac{3\pi}{4} \cdot \cos\frac{\pi}{6} - \sin\frac{3\pi}{4} \cdot \sin\frac{\pi}{6}$

$= -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$ or $\frac{-\sqrt{6}-\sqrt{2}}{4}$

2. $\cos\left(\frac{19\pi}{16}\right)\cos\left(\frac{7\pi}{16}\right) + \sin\left(\frac{19\pi}{16}\right)\sin\left(\frac{7\pi}{16}\right) =$

$= \cos\left(\frac{19\pi}{16} - \frac{7\pi}{16}\right)$

$= \cos\left(\frac{12\pi}{16}\right) = \cos\left(\frac{3\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$

More challenging example:

Remember: SOH-CAH-TOA
Pythagorean Thm.

Given: $\cos\alpha = \frac{3}{5},\ 0 < \alpha < \frac{\pi}{2}$

$\sin\beta = -\frac{1}{4},\ \pi < \beta < \frac{3\pi}{2}$

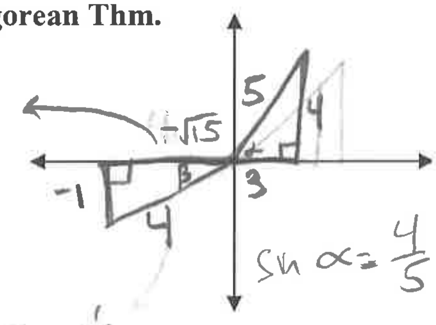
$(-1)^2 + b^2 = 4^2$

$b^2 = 15$

$b = \sqrt{15}$

$\downarrow$

$-\sqrt{15}$ since we're going left.



$\sin\alpha = \frac{4}{5}$

Find: $\cos(\alpha + \beta)$.

$= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

So, $\cos\beta = -\frac{\sqrt{15}}{4}$

$= \frac{3}{5} \cdot \frac{-\sqrt{15}}{4} - \frac{4}{5} \cdot \left(-\frac{1}{4}\right)$

$= \frac{-3\sqrt{15}}{20} + \frac{4}{20} = \boxed{\frac{-3\sqrt{15} + 4}{20} \text{ or } \frac{4 - 3\sqrt{15}}{20}}$

Prove the following are equal. ***Hint: Use formulas and Unit Circle. Only work on the left side.***

3. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$= \cos\frac{\pi}{2} \cdot \cos x + \sin\frac{\pi}{2} \cdot \sin x$

$= 0 \cdot \cos x + 1 \cdot \sin x$

$= \boxed{\sin x}$ ✓

4. $\cos\left(x + \frac{3\pi}{2}\right) = \sin x$

$= \cos x \cdot \cos\frac{3\pi}{2} - \sin x \cdot \sin\frac{3\pi}{2}$

$= \cos x(0) - \sin x(-1)$

$= 0 + \sin x$

$= \boxed{\sin x}$ ✓